A Method for Deriving Transverse Masses Using Lagrange Multipliers

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Abstract
We use Lagrange multipliers to extend the traditional definition of Transverse Mass used in experimental high energy physics. We demonstrate the method by implementing it to derive a new Transverse Mass that can be used as a discriminator to distinguish between top decays via a charged W or a charged Higgs Boson.

1 Introduction

If a particle decay products are not fully measurable due to the presence of Neutrinos in the final states or energy flowing in the direction of the beam pipe, it is impossible to fully reconstruct the mass of the decaying particle. In proton-proton collisions, the missing momentum in the z direction is a complete unknown and cannot be deduced by energy-momentum balance considerations (unlike the missing transverse momentum). However, using known constraints on unmeasurable physical quantities (usually from energy-momentum conservation) it is possible to find an upper or lower bound on the decaying particle mass. In section 3 we derive the classical W transverse mass when the W Boson decays to a Lepton and a Neutrino, $W \to \ell\nu$ [1][2]. The W mass bounds this Transverse mass from above. In section 4 we derive a different expression for the W "Transverse Mass" in top decays involving multi-neutrinos final states, $t \to Wb \to \tau\nu b \to \ell\nu\bar{\nu}\nu b$. Here the mass is bounded from below. We then implement the same expression to discriminate a top decaying via a W and a charged Higgs Boson. Charged Higgs Bosons are smoking guns for the existence of theories beyond the Standard Model, like the Minimal Supersymmetric extensions of the Standard Model (MSSM) which contain two charged Higgs Bosons [3]. In section 5 we demonstrate how to implement the method to derive additional kinematical bounds. We then conclude.

2 Notation

For convenience we adopt the following notation:

p is a 4-vector

$$p \equiv (\vec{p_{\parallel}}, \vec{p_T}) \tag{1}$$

with

$$\vec{p_T} = (p_x, p_y) \tag{2}$$

and

$$\vec{p_{\parallel}} = (E, p_z) \tag{3}$$

satisfying

$$p^2 = m^2 = \vec{p_{\parallel}}^2 - \vec{p_T}^2 \tag{4}$$

$$\vec{p_{\parallel}}^2 = E^2 - p_z^2 \tag{5}$$

$$\vec{p_T}^2 = p_x^2 + p_y^2. \tag{6}$$

The missing momentum is denoted by $\not{p}$.

3 The case of one Neutrino: deriving the classical W transverse mass

Suppose the outcome of a proton-proton collision is a W Boson decaying into a lepton and a Neutrino, $W \to \ell\nu$. The mass of the W is given by $M^2 = (p_\ell + \not{p})^2$. Assuming one can measure the transverse missing momentum, we consider $\not{p}^Z$ and $\not{E}$ as two unknown quantities satisfying the constraint $\not{p}^2 = 0$. Using Eq. 4 the mass can be expressed as

$$M^2 = (\vec{p}_{\parallel\ell} + \vec{\not{p}}_{\parallel})^2 - (\vec{p}_{T\ell} + \vec{\not{p}}_T)^2 \tag{7}$$

and the constraint as

$$\vec{\not{p}}_{\parallel}{}^2 - \vec{\not{p}}_T{}^2 = 0. \tag{8}$$

To find an extremum to the squared mass (7) under the constraint (8) we use the method of Lagrange multipliers. Since the transverse momenta are orthogonal to the parallel momenta we find

$$\frac{\partial}{\partial \vec{\not{p}}_{\parallel}}((\vec{p}_{\parallel\ell} + \vec{\not{p}}_{\parallel})^2 - \lambda(\vec{\not{p}}_{\parallel})^2) \mid_{\lambda=\lambda_0, \vec{\not{p}}_{\parallel} = \vec{\not{p}}_{\parallel 0}} = 0 \tag{9}$$

We find

$$(\vec{p}_{\parallel\ell} + \vec{\not{p}}_{\parallel 0}) = \lambda_0 \vec{\not{p}}_{\parallel 0}. \tag{10}$$

Using (8) the solution λ_0 for λ is

$$(\lambda_0 - 1)^2 = \frac{\vec{p}_{\parallel\ell}{}^2}{\vec{\not{p}}_T{}^2} \tag{11}$$

Approximating the lepton to be massless $(\vec{p}_{\parallel\ell}{}^2 = \vec{p}_{T\ell}^2)$, we find

$$\lambda_0 = 1 + \frac{p_{T\ell}}{\not{p}_T} \tag{12}$$

where $p_{T\ell}$ and $\not{p}_T$ are the magnitudes of the transverse lepton and missing momenta. Plugging λ_0 into (10), and using (8) we find the extremum for the squared mass (Eq. 7) denoted by M_T. It is given by

$$M_T^2 = (\not{p}_T + p_{T\ell})^2 - (\vec{p}_{T\ell} + \vec{\not{p}}_T)^2 \tag{13}$$

which can also be expressed as

$$M_T^2 = 2\ \not{p}_T p_{T\ell}(1 - cos\psi) \tag{14}$$

where ψ is the angle between the lepton and the missing momentum in the transverse plane. Equations 13 and 14 are the familiar forms of the W transverse mass [1][2].

Figure 1 shows the distribution of the W transverse mass, using partons generated with Pythia [4]. One can clearly see that this distribution has a sharp upper bound at $M_T = M_W$, as expected.

4 The case of multi-neutrinos: deriving a Chraged Higgs transverse mass

Here we consider a final state with a lepton and three Neutrinos. Such a final state can occur in top decays. For example $t \to Wb \to \tau\nu b \to \ell\nu\bar{\nu}\nu b$. One interesting case is when the W Boson is replaced by the charged Higgs Boson H^+, which occurs in Mimnimal Supersymmetric extensions of the Standard Model (MSSM). Here one would like to tell W mediated decays from Higgs mediated decays. One possible discriminating variables is the transverse mediating particle mass which is derived in this section.

The difference between this case and the previous one (section 3) is that here the missing energy-momentum is a result of a few Neutrinos and one cannot use the constraint (Eq. 8). Here $\not{p}^2 = \vec{\not{p}}_{\parallel}{}^2 -$

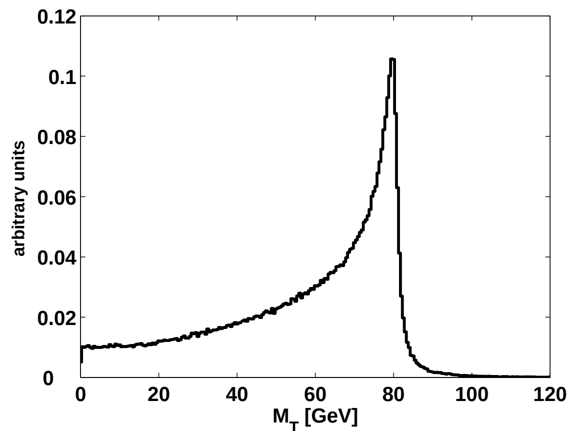

Fig. 1: The W transverse mass distribution

$\vec{\not{p}}_T{}^2 \neq 0$. However, this constraint can be replaced by a constraint given by the parent particle mass, i.e. the top quark mass

$$M_{top}^2 = (p_\ell + \not{p} + p_b)^2 \tag{15}$$

where p_b is the bottom-quark 4-momentun.

The mediating particle mass which is subject to minimization under the constraint (15) is given by

$$M^2 = (p_\ell + \not{p})^2. \tag{16}$$

Repeating the same procedure as in section 3 we find

$$(\vec{p}_{\parallel\ell} + \vec{\not{p}}_{\parallel 0}) = \lambda_0(\vec{p}_{\parallel\ell} + \vec{\not{p}}_{\parallel 0} + \vec{p}_{\parallel b}). \tag{17}$$

Where λ_0 and $\vec{\not{p}}_{\parallel 0}$ are minimizing Eq. 16. With the help of the constraint (15) we find the extremum denoted by M_T,

$$M_T^2 = \left(\sqrt{M_t^2 + (\vec{p}_{T\ell} + \vec{p}_{Tb} + \vec{\not{p}}_T)^2} - p_{Tb}\right)^2 - \left(\vec{\not{p}}_T + \vec{p}_{T\ell}\right)^2 \tag{18}$$

where p_{Tb} is the scalar magnitude of the bottom quark transverse momentum.

Figure 2 shows the parton distribution of this transverse mass (Eq. 18) for two cases. A mediating W Boson (full line) and a mediating Charged Higgs Boson with a hypothetical mass of 130 GeV/C^2. One can clearly see the power of this method to distinguish between the two possible top-decays. Note also that in this case the distributions have a threshold and a Jacobian peak at the mediating particle mass (the W or the charged Higgs mass).

5 The general case: other applications

In the general case let $F(\vec{\not{p}}_\parallel, \vec{\not{p}}_T, ...)$ be a kinematical variable which is a function of the unknown $\vec{\not{p}}_\parallel$ and other known variables. Let $f(\vec{\not{p}}_\parallel, \vec{\not{p}}_T, ...) = 0$ be a constrained on the variables involved. Define

$$G(\lambda, \vec{\not{p}}_\parallel, \vec{\not{p}}_T, ...) = F(\vec{\not{p}}_\parallel, \vec{\not{p}}_T, ...) + \lambda f(\vec{\not{p}}_\parallel, \vec{\not{p}}_T, ...) \tag{19}$$

Solve for λ and $\vec{\not{p}}_\parallel$ that minimizes or maximizes $G(\lambda, \vec{\not{p}}_\parallel, \vec{\not{p}}_T, ...)$, i.e.

$$\frac{\partial}{\partial \vec{\not{p}}_\parallel} G(\lambda, \vec{\not{p}}_\parallel, \vec{\not{p}}_T, ...)|_{\lambda=\lambda_0, \vec{\not{p}}_\parallel = \vec{\not{p}}_{\parallel 0}} = 0 \tag{20}$$

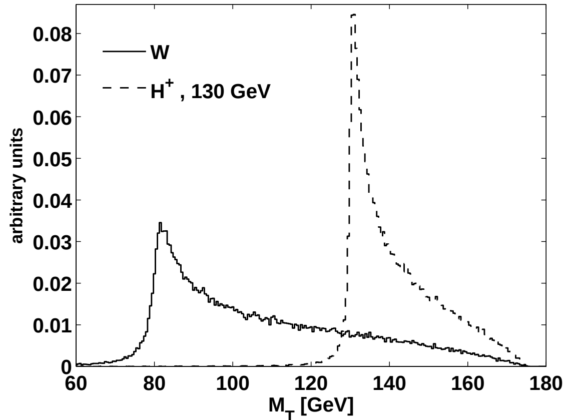

Fig. 2: The new transvesre mass distribution for a mediating W Boson (full line) and a 130 GeV Charged Higgs Boson (dashed line)

The solution gives a relation

$$g(\lambda_0, \vec{\not p}_{\parallel 0}, \vec{\not p}_T, ...) = 0 \tag{21}$$

The resulting "transverse" F is then

$$F_T = F(\vec{\not p}_{\parallel 0}, \vec{\not p}_T, ...) \tag{22}$$

6 Conclusion

We have shown how to construct discriminators useful for Hadron colliders where only the transverse missing energy can be measured. These discriminators are an extension of the traditional W transverse mass used in High Energy Physics and we derive them using Lagrange multipliers.

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